Applications of Derivatives: Displacement, Velocity and Acceleration

Kinematics is the study of motion and is closely related to calculus. Physical quantities describing motion can be **related** to one another **by derivatives**.

Below are some quantities that are used with the application of derivatives:

1. **Displacement** is the shortest distance between two positions and has a direction.
   Examples:
   - The park is 5 kilometers north of here
   - $x(t)=5t$, where x is displacement from a point P and t is time in seconds

2. **Velocity** refers to the speed and direction of an object.
   Examples:
   - Object moving 5 m/s backwards
   - $v(t) = t^2$, where v is an object's velocity and t is time in seconds

3. **Acceleration** is the rate of change of velocity per unit time. Imagine increasing your speed while driving. Acceleration is how quickly your speed changes every second.

Examples:

- Increasing speed from 10 m/s to 25 m/s in 5 s results in:

$$\text{Acceleration} = \frac{25\ m/s - 10\ m/s}{5\ s} = 3\ m/s^2$$

$a(t) = -t$. where a is an object's acceleration and t is time in seconds

Displacement, velocity and acceleration can be expressed as **functions of time**. If we express these quantities as functions, they can be **related by derivatives**.
Given $x(t)$ as displacement, $v(t)$ as velocity and $a(t)$ as acceleration, we can relate the functions through derivatives.

$$\boldsymbol{a(t) = v'(t) = x''(t)}$$

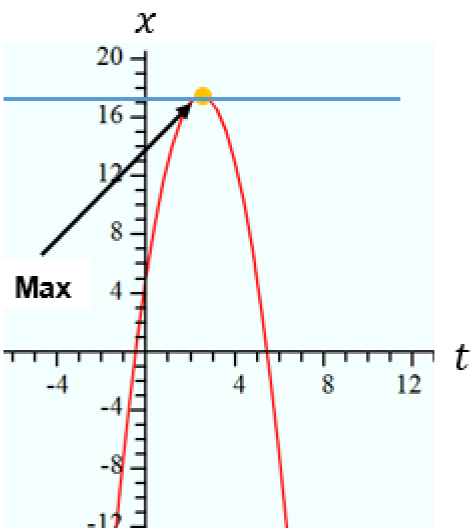
Equivalently, using Leibniz notation:

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

The **maximum** of a motion function occurs when the **first derivative** of that function equals 0.

For example, to find the time at which **maximum displacement** occurs, one must equate the **first derivative of displacement (i.e. velocity) to zero**.

Notice on the right-hand graph, the maximum of the displacement function, *x(t)*, occurs along the flat blue line where the rate of change is zero.



Example 1

If a particle is moving in space with a velocity function, $v(t)=t^2-2t-8$ where *t* is in seconds and velocity is measured in meters per second:

a) At what time(s), if any, is the particle at rest?
b) What is the acceleration of the particle at t=3 seconds?

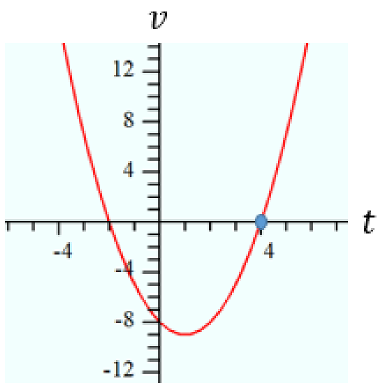
Solution:

a) If the particle is at rest, $v(t)=0$ (velocity is zero at rest)

Solving for t when v(t) = 0:

$t^2 - 2t - 8 = 0$
$(t - 4)(t + 2) = 0$
$t = 4$ or $t = -2$

Since negative time is **impossible**, the only time at which the particle is at rest is 4 seconds.



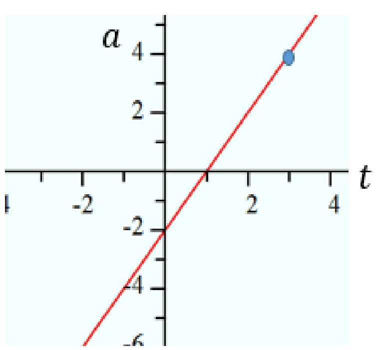
b) First find the **function for acceleration** by taking the derivative of velocity.

$a(t) = v'(t)$
$a(t) = 2t - 2$

Substitute t = 3 s in the acceleration function:

$a(3) = 2(3) - 2 = 4\ m/s^2$

Thus, the acceleration at t = 3 s is 4 m/s^2.



Example 2

A soccer ball is kicked into the air so that the path of its flight can be modeled by the function, where *t* is in seconds and x is meters **above ground**:

$$x(t) = -4.9t^2 + 9.8t + 5$$

a) At what time will the ball land?
b) How many meters above ground was the ball kicked?
c) What is the maximum height the ball will reach and at what time will this occur?
d) What is the acceleration (with direction) of the ball at $t=3\ s$?

Solution:

a) Since $x(t)$ models height above ground, $x(t)=0$ when the ball hits the ground

Solving for t when x(t) = 0:

$0 = -4.9t^2 + 9.8t + 5$

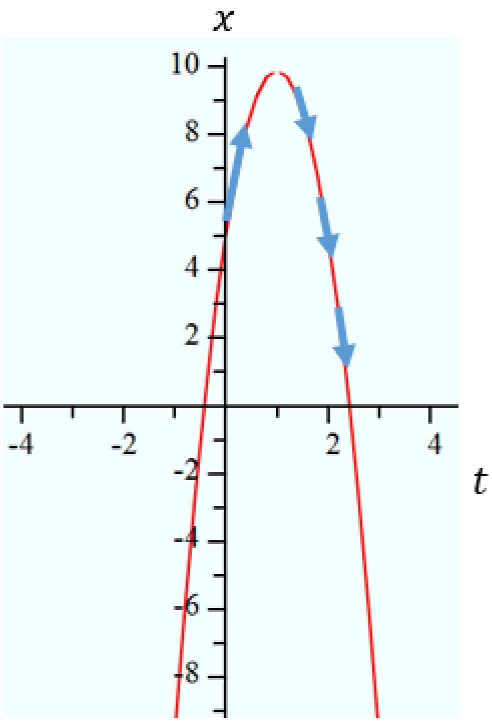
Since this equation cannot be factored, the quadratic equation must be used.

$$t = \frac{-9.8 \pm \sqrt{9.8^2 - 4(-4.9)(5)}}{2(-4.9)}$$

$$= \frac{-9.8 \pm \sqrt{194.04}}{-9.8}$$

$t = 2.421$ s or $t = -0.421$ s (to 3 decimal places)

However, t is greater than 0, (since time cannot be negative). Thus, the ball hits the ground 2.421 seconds after being launched.



b) The **initial height above ground occurs when t = 0.** Substitute t = 0 into *x(t)*:

$x(0) = -4.9(0)^2 + 9.8(0) + 5 = 5$

Thus, the ball is thrown from 5 meters above ground.

c) Maximum height occurs when the first derivative equals zero.

Find the first derivative:
x'(t) = -9.8*t* + 9.8

Solve for *t* when x'(t) = 0, time when the ball reaches maximum height:
0 = -9.8*t* + 9.8
-9.8*t* = -9.8
t = 1 s

Substitute t = 1 s into x(t):
$x(1) = -4.9(1)^2 + 9.8(1) + 5 = 9.9$ m

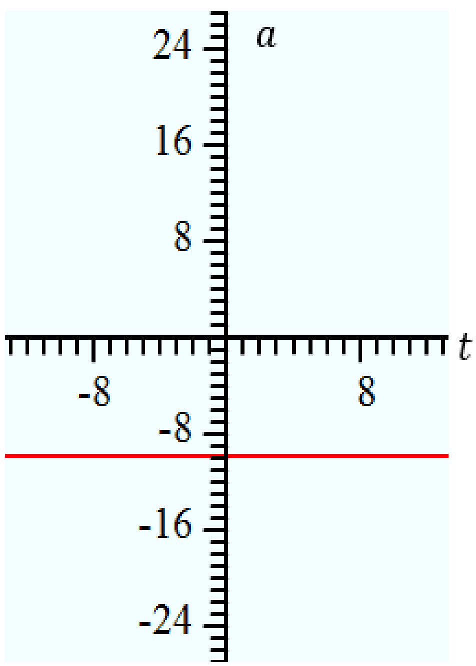
Thus, the maximum height is 9.9 m.

d) Acceleration is equal to the second derivative of displacement.

Finding second derivative:
x''(t) = -9.8

Acceleration is constant for all values of time, *t*. Thus, **x''(3) = -9.8**.

Thus, the acceleration of the ball at 3 seconds is 9.8 m/s^2 [down].
The negative implies that the acceleration is **downward**. The acceleration of the ball equals the acceleration of gravity: **9.8 m/s^2 [down]**. This is because the ball is subject to gravity at **all times** during its flight.



Exercises:

Problem 1:

If a particle moves in space according to the function $x(t) = t^3 - 4t^2$, where t is time in seconds and x is displacement from the origin in centimeters (with positive to the right):

a) Find the acceleration of the particle at $t = 2\ s$.
b) Determine at what displacement(s) from the origin the particle is at rest.
c) Find the maximum velocity of the particle.

Problem 2:

An electron moves such that its velocity function with respect to time is $v(t)=e^{2t-2}$, where t is time in seconds and v is velocity in meters per second:

a) What is the acceleration of the electron at $t = 10\ s$?
b) Is the electron ever at rest? Algebraically explain why or why not.

Problem 3:

A ball is thrown in the air and follows the displacement function $x(t) = -4.9t^2 + 4.9t + 9.8$, where t is time in seconds and x is displacement above the ground in meters:

a) What is the initial height (above ground) from which the ball is thrown?
b) At what time does the ball reach its maximum height? What is the maximum height above ground?
c) Determine when the ball hits the ground?
d) What is the acceleration of the ball at $t = 1\ s$, $t = 1.5\ s$ and $t = 2s$? What do you notice?

Solutions:

1a) 4 m/s^2 [right]:

1b) At origin and 256/27 cm [left of origin]

1c) 16/3 m/s^2 [left]

2a) $2e^{18}$ m/s^2

2b) Never, $e^{2t-2}=0$ has no solution

3a) 9.8 m;

3b) t = 0.5 s and x(0.5)=11.025 m

3c) t = 2s;

3d) -9.8 m/s^2 (constant due to gravity)